

A NOTE ON NON-NEGATIVELY CURVED BERWALD SPACES

MARTIN KELL

ABSTRACT. In this note it is shown that Berwald spaces admitting the same norm-preserving torsion-free affine connection have the same (weighted) Ricci curvatures. Combing this with Szabó's Berwald metrization theorem one can apply the Cheeger-Gromoll splitting theorem in order to get a full structure theorem for Berwald spaces of non-negative Ricci curvature. Furthermore, if none of the factor is a symmetric space one obtains an explicit expression of Finsler norm of the resulting product. By the general structure theorem one can apply the soul theorem to the factor in case of non-negative flag curvature to obtain a compact totally geodesics, totally convex submanifolds whose normal bundle is diffeomorphic to the whole space.

In the end we given applications to the structure of Berwald-Einstein manifolds and non-negatively curved Berwald spaces of large volume growth.

In geometry and analysis, curvature is an important tool that rules regularity of solution of PDEs and helps to classify spaces. The two main curvature indicators are sectional and Ricci curvature. Assuming global bounds on those quantities one can show that a manifold must have certain topological type. Cheeger and Gromoll managed [CG71] to show that a Riemannian manifold with non-negative Ricci curvature contains an isometrically embedded if and only if it isometrically splits as a Cartesian product of another manifold of non-negative Ricci curvature and a line. Later Perelman showed [Per94] that Riemannian manifold of non-negative sectional curvature diffeomorphic to the normal bundle of an embedded compact totally geodesic submanifold. This submanifold is called soul and every other such submanifold must be isometric.

On Finsler manifold, i.e. those whose tangent spaces are not Euclidean spaces, several notions of curvature exist. The most prominent being flag and Ricci curvature. There are result on constantly curved space and strictly positively curved spaces (see e.g. [She96, She02]). Only recently inspired by the theory of weighted Ricci curvature bounds Ohta [Oht13] managed to show a diffeomorphic splitting theorem for general Finsler manifolds of non-negative Ricci curvature.

A subclass of Finsler manifolds are Berwald spaces. Whereas general Finsler manifolds can have non-isometric tangent spaces, Berwald spaces are modeled on the “same” tangent space. An important property of those spaces is that the parallel transport is linear and admits a unique affine connection. In [Sza81] (see also [Sza06]) Szabó managed to show that there is indeed a Riemannian metric whose Levi-Civita connection agrees with the affine connection of the Berwald space. In particular, geodesics in a Berwald space are affinely equivalent to geodesics of a

Date: February 25, 2015.

During the research the author was funded by an EPDI-fellowship at the IHÉS. He wants to thank the IHÉS for its hospitality and financial support. Thanks also to J. Szilasi for suggesting a more direct approach using only the affine connection of the Berwald space.

Riemannian manifold. As the theory of Levi-Civita connections and their structure is well-known, in particular their product structure (de Rham Decomposition Theorem [dR52]) and their rigidity (Berger-Simmons Theorem, see [Olm05] for a new geometric proof), one can exactly tell which Levi-Civita connections can be metrized via a non-Riemannian Berwald metric.

Because of their nicely behaved connections, there are some results only known to hold for Berwald spaces, but not (yet) for general Finsler spaces, e.g. Busemann convex Berwald spaces are exactly the ones having non-positively flag curvature [KK06]. Furthermore, for Berwald spaces of non-negative Ricci curvature, Ohta [Oht13] proved an extended diffeomorphic splitting theorem, i.e. he showed that a Berwald space has a maximal flat factor and the fibers over this factor are totally geodesic.

In this article we first show that Ricci curvature only depends on the induced connection and thus is an affine invariant of Berwald spaces. As Szabó showed that every Berwald space is affinely equivalent to a Riemannian manifold, one can apply the classical Cheeger-Gromoll splitting. This together with the general structure theorem of Berwald space yields. Note that this calculation only needs the Berwald Metrization Theorem and results from Riemannian geometry. In particular, the de Rham Decomposition Theorem.

Theorem. *A geodesically complete simply connected Berwald space (M, F) of non-negative Ricci curvature has the following form*

$$M = M_0 \times \cdots \times M_n$$

where $M_0 = \mathbb{R}^k$ is the maximal flat factor and each M_i is simply connected and either an affinely rigid Riemannian manifold or a higher rank symmetric Berwald space of compact type. In particular, each factor has non-negative Ricci curvature.

Furthermore, if none of the M_i is a higher rank symmetric Berwald space then the Finsler norm has the following form

$$F(v_0, \dots, v_n) = G(v, F_1(v_1), \dots, F_n(v_n))$$

where $v_i \in TM_i$, F_i is a norm induced by a unique up to scale Riemannian metric g_i on M_i and G is a Minkowski norm on \mathbb{R}^{k+n} which is symmetric in the last n -coordinates.

Let g_i denote either the unique Riemannian metric on M_i or a Riemannian metric on the symmetric space M_i such that (M_i, g_i) is Einstein, e.g. $g_i = \text{Ric}_i$. Note that in both cases (M_i, F_i) and (M_i, g_i) are affinely equivalent. Then the Cartesian product (M, g) of (M_i, g_i) with Riemannian metric $g = \sum g_i$, where g_0 is any flat metric on M_0 , has non-negative sectional curvature if M has non-negative flag curvature. Furthermore, (M, F) and (M, g) are affinely equivalent, i.e. their geodesics agree. If one applies the soul theorem [Per94] to (M, g) then the soul S is also a compact, totally geodesic submanifold for the Berwald space (M, F) . In particular, M is diffeomorphic to the normal bundle of S .

Note. An earlier version of this paper relied heavily on [Oht13] and its use of the Chern connection. The author wants to thank Szilasi for referring him to the paper [DKY15] and suggesting to avoid complicated calculation involving the Chern connection.

1. AFFINE CONNECTIONS AND THE DE RHAM DECOMPOSITION

Let M be a connected, n -dimensional smooth manifold and denote by $C^\infty(M, TM)$ be the space of vector fields.

Definition 1 (Affine connection). An *affine connection* is a bilinear map

$$\begin{aligned} \nabla : C^\infty(M, TM) \times C^\infty(M, TM) &\rightarrow C^\infty(M, TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

such that for all smooth function f and all vector field X, Y on M

- ($C^\infty(M, \mathbb{R})$ -linearity) $\nabla_{fX} Y = f\nabla_X Y$
- (Leibniz rule) $\nabla_X(fX) = df(X)Y + f\nabla_X Y$.

We say that ∇ is *torsion-free* if for all vector field X and Y

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Remark. Note that every connection has an associated covariant derivative which agrees with the connection when applied to vector fields. In the Finsler setting the gradient of functions, usually denoted by ∇ as well, is defined via Legendre transform and does not agree with the connection or the covariant derivative applied to f . Therefore, it is more convenient to use D as a symbol for the covariant derivative.

Suppose the tangent bundle TM splits into a direct sum $V \oplus W$ such $\nabla_X(Y + Z) = \nabla_X Y$ whenever $X, Y \in V$ and $Z \in W$ or $X, Y \in W$ and $Z \in V$. Then ∇ can be written as a sum of two affine connections only acting on V and W respectively. In this case one says that ∇ is reducible. If there does not exist such a splitting then ∇ is said to be irreducible.

In local coordinates one can define the coefficients Γ_{ij}^k of the connection as follows

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k$$

where $\{\partial_i\}$ spans is a local trivialization of the tangent bundle. Then the covariant derivative D acting on vector fields is defined as

$$D_V X = \sum_{i,j=1}^n \left\{ v^i \partial_j X^i + \sum_{k=1}^n \Gamma_{jk}^i v^j X^k \right\} \partial_i$$

where $X = \sum_{i=1}^n X^i \partial_i$ and $V = \sum_{i=1}^n v^i \partial_i$.

If (M, g) is a Riemannian manifold then there is a unique torsion-free affine connection, called *Levi-Civita connection*, which is also metric compatible, i.e.

$$\nabla g = 0.$$

This is equivalent to require the connection coefficients to have the following form

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n g^{km} (\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij})$$

where $g = (g_{ij})$ and (g^{ij}) is the inverse of (g_{ij}) . In the Riemannian case the coefficients are usually called Christoffel symbols.

For a simply connected Riemannian manifold which is geodesically complete (see section on Finsler structures below) one can split the manifold into a product of

irreducible components, i.e. those having irreducible Levi-Civita connection. The following theorem is well-known.

Theorem 2 (de Rham Decomposition). *Let M be a simply connected, geodesically complete Riemannian manifold with Levi-Civita connection ∇ . Then M is isometric to a product, i.e.*

$$M = M_0 \times \cdots \times M_n$$

metric $g = \sum g_i$ where (M_0, g_0) is isometric to an Euclidean space and each (M_i, g_i) admits a unique irreducible Levi-Civita connection ∇_i . Furthermore, the Levi-Civita connection ∇ can be written as a sum of the ∇_i .

Every affine connection induces a unique linear *parallel transport* along curves, i.e. for a smooth curve there is a map

$$P_\gamma : T_{\gamma_0} M \rightarrow T_{\gamma_1} M$$

such that $P_\gamma v$ is the unique vector X_{γ_1} of a vector field X such that $\nabla_{\gamma_t} X = 0$ for all $t \in [0, 1]$. It is easy to see that P_γ is an invertible linear map between vector spaces. We define the holonomy group as those invertible linear maps $A : T_x M \rightarrow T_x M$ for which there is a curve γ with $\gamma(0) = \gamma(1) = x$ and $A = P_\gamma$. Note that if ∇ is a Levi-Civita connection then the parallel transport preserves the norm of a vector. In particular, every $A \in \mathcal{H}_x$ is an orthogonal transformation on $(T_x M, g_x)$.

Theorem 3 (Berger-Simmons). *Let (M, g) be a simply connected, geodesically complete, irreducible Riemannian manifold. If the holonomy does not act transitively on the unit sphere then (M, g) is a Riemannian symmetric space.*

It is not important to know the exact definition of symmetric spaces. We only need the following facts: every symmetric space has a unique rank which is a natural number ≥ 1 . And every higher rank symmetric space is either non-compact or compact and embeds a totally geodesic flat submanifold of dimension equal to its rank. In case the symmetric space is non-compact and irreducible this flat submanifold is isometric to \mathbb{R}^n , in particular, there exists an isometrically embedded line.

Lemma 4 (Holonomy orbits in rank 1). *Assume (M, g) is a symmetric space of rank 1. Denote by \mathcal{H}_x the holonomy group of the Levi-Civita connection acting on $T_x M$ then for $\mathcal{H}_x v \cap V \neq \emptyset$ for every $v \in T_x M$ and one dimensional subspace V of $T_x M$. In words, the orbit of the holonomy intersects every one dimensional subspace.*

Proof. This is a well-known fact from the theory of symmetric spaces. Just note that every one dimensional subspace of $T_x M$ is a Cartan subalgebra and every orbit of the holonomy group intersects this algebra, see [Sza06, Lemma 3.1]. \square

2. FINSLER STRUCTURES AND THE CHERN CONNECTION

Let M be a connected, n -dimensional C^∞ -manifold.

Definition 5 (Finsler structure). A C^∞ -Finsler structure on M is a function $F : TM \rightarrow [0, \infty)$ such that the following holds

- (1) (Regularity) F is C^∞ on $TM \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ stands for the zero section,
- (2) (Positive homogeneity) for any $v \in TM$ and any $\lambda > 0$, it holds $F(\lambda v) = \lambda F(v)$,

(3) (Strong convexity) In local coordinates $(x^i)_{i=1}^n$ on $U \subset M$ the matrix

$$(g_{ij}(v)) := \left(\frac{1}{2} \frac{\partial^2(F^2)}{\partial v^i \partial v^j}(v) \right)$$

is positive-definite at every $v \in \pi^{-1}(U) \setminus 0$ where $\pi : TM \rightarrow M$ is the natural projection of the tangent bundle.

Strictly speaking, this is nothing more than defining a Minkowski norm $F|_{T_x M}$ on each $T_x M$ with some regularity requirements depending on x . We don't require F to be absolutely homogeneous, i.e. $F(v) \neq F(-v)$ is possible. In such a case the "induced" distance is not symmetric.

If each F_x is induced by an inner product g_x then the Finsler structure is actually a Riemannian structure and (M, g) a Riemannian manifold. Note that in that case $(g_{ij}(v_x)) = g_x$ and the distance is symmetric.

A geodesic from x to y is a curve $\gamma : [0, 1] \rightarrow M$ which minimizes the following functional

$$\gamma \mapsto \int_0^1 F(\dot{\gamma}_t) dt.$$

Note that in general that the reversed curve $\bar{\gamma}(t) = \gamma(1 - t)$ is not a geodesic from x to y . Then there is a (possibly asymmetric) metric d defined as

$$d(x, y) = \inf_{\gamma} \int_0^1 F(\dot{\gamma}_t) dt$$

where the infimum is taken over all curves $\gamma : [0, 1] \rightarrow M$ connecting x and y .

3. BERWALD SPACES AND AFFINE RIGIDITY

In contrast to the Riemannian setting there is no unique (affine) connection on a general Finsler manifolds. However, there is a subclass containing the Riemannian manifolds where this is the case.

Definition 6 (Berwald space). A Finsler structure F on M is called Berwald if it admits a (unique) torsion-free affine connection whose induced parallel transport preserves the Finsler norm.

Remark. In this note we avoid the use of the Chern connection. Usually one requires the Chern connection to be an affine connection, i.e. independent of the reference vector. The opposite statement is shown in [SLK11], i.e. a torsion-free, norm preserving affine connection is the Chern connection and the space is Berwald.

Note that for general functions $G : TM \rightarrow \mathbb{R}$ one can interpret $\nabla G = 0$ as being invariant under parallel transport induced by ∇ . Hence for Berwald spaces might say that $\nabla F = 0$ is metric compatibility similar to the Riemannian case.

It is well-known that every geodesic satisfies the following

$$D_{\dot{\gamma}} \dot{\gamma} = 0$$

and $F(\dot{\gamma}) \equiv \text{const}$, i.e. it is a constant speed auto-parallel curve. Also the converse holds; every auto-parallel curve is locally a geodesic, i.e. it is locally the distance minimizing curve between two points. We say that a (M, F) is forward geodesically complete if the space is complete and any auto-parallel curve $\gamma : [0, t] \rightarrow M$ with $D_{\dot{\gamma}} \dot{\gamma} = 0$ can be extended to an auto-parallel curve beyond t .

In particular, if a connection preserves two different Finsler norms F_1 and F_2 on M then their auto-parallel curves agree and their geodesics are the same, i.e. γ is auto-parallel w.r.t. the first space iff it is w.r.t. the second space; geodesics only differ in their speed. In such a case we say that the two spaces are *affinely equivalent*.

Lemma 7. *Let F_1 and F_2 be two Berwald structure on M such that their induced connections agree. Then they are affinely equivalent.*

Because on a Berwald space F and the reversed Finsler structure $\bar{F}(v) = F(-v)$ have the same induced connections, a reversed geodesic is also auto-parallel and therefore they are (local) geodesics as well. In particular, a forward geodesically complete Berwald space is necessary backward geodesically complete. In such a case we just say the Berwald space is geodesically complete.

An important ingredient of the Berwald classification theorem was the following result.

Theorem 8 (Berwald metrization [Sza81]). *If (M, F) is a Berwald space with induced connection ∇ then there exist (uniquely defined) Riemannian metric g whose Levi-Civita connection is ∇ . In particular, (M, F) and (M, g) are affinely equivalent.*

Note there might be several Riemannian metrics compatible with ∇ . However, the metric g is intrinsically defined and if (M', F') is isometric to (M, F) then also (M, g) and (M', g') .

Szabó actually showed only certain connections admit non-Riemannian Berwald structures. His idea was to use the fact that an affine connection induces a uniquely defined notion of parallel transport on the tangent bundle. In particular, the Finsler norm F_x at $T_x M$ needs to be invariant by the holonomy group \mathcal{H}_x . However, this is a rather rigid condition.

Definition 9 (Affine rigidity). A Berwald space (M, F) is affinely rigid if for every other Berwald space (M', F') affinely equivalent to (M, F) , i.e. their induced connection agree, it holds $F = \lambda F'$ for some $\lambda > 0$.

Because every Berwald space is affinely equivalent to a Riemannian manifold, the only affinely rigid Berwald spaces are Riemannian manifolds. Now from the Berger-Simmons theorem and their corollaries we get the following affine rigidity theorem.

Theorem 10 (Affine rigidity classification). *Assume (M, F) is an irreducible geodesically complete Berwald space. Then (M, F) is either a affinely rigid Riemannian manifold or a higher rank symmetric Berwald space. Furthermore, every continuous function $G : TM \rightarrow \mathbb{R}$ which is invariant under parallel transport has the following form*

$$G(v) = \varphi(F(v))$$

for some $\varphi : [0, \infty) \rightarrow \mathbb{R}$.

Proof. Let \mathcal{H}_x be the holonomy group of the connection acting on $(T_x M, F_x)$. If \mathcal{H}_x acts transitively on the unit sphere then any function G is uniquely determined by the length of the vector, i.e. $G(v) = \varphi(F(v))$. In particular, F_x is a norm coming from an inner product, i.e. (M, F) is Riemannian. Now let (M, F') be another Berwald space with the same connection. Then F' is preserved by parallel

transport and therefore $F' = \varphi \circ F$. By positive 1-homogeneity we see that $F' = \lambda F$, i.e. (M, F) is affinely rigid.

If the holonomy group does not act transitively then (M, ∇) is a symmetric space. If (M, ∇) is a higher rank symmetric space then there is nothing to prove. So assume (M, ∇) has rank 1. Note that M admits a Riemannian metric g with Levi-Civita connection ∇ .

Let G be a continuous function invariant under parallel transport. In particular, it is invariant under the holonomy group. Then Lemma 4 implies that G is uniquely determined by its restriction to a one dimensional subspace V of $T_x M$. Indeed, for every $w \in T_x M$ there is an $h \in \mathcal{H}_x$ such that $hw = v \in V$ and $G(w) = G(v)$. It remains to show that $G(v) = G(-v)$.

Assume by contradiction $G(v) \neq G(-v)$. Then the following sets are two disjoint open subsets covering the $S_x M$ w.r.t. the Riemannian metric g :

$$\begin{aligned} A_< &= \{w \in S_x M \mid G(w) < \frac{G(v) + G(-v)}{2}\} \\ A_> &= \{w \in S_x M \mid G(w) > \frac{G(v) + G(-v)}{2}\}. \end{aligned}$$

However, $S_x M$ is connected and cannot be covered by disjoint open sets. Therefore, $G(v) = G(-v)$, i.e. $G(v) = \varphi(g(v))$. Similar to the transitive case, F must be induced by a Riemannian metric and any other Berwald norm must be a multiple of F . \square

In case the function $G(v) = F(v, v')$ where (v, v') is a tangent vector on $M = M_1 \times M_2$ and F a Finsler structure, the theorem has the following interpretation: Fix a vector $v' \in (TM_2)_y$ with $F(0, v') < 1$. Then

$$A_{v'} = \{v \in (TM_1)_x \mid F(v, v') \leq 1\}$$

is strictly convex in $TM_{(x,y)}$ with smooth boundary containing the origin of TM_1 in its interior. Thus it represents a Finsler norm on TM_1 . If (M_1, F_1) is affinely rigid this norm must be the unique Finsler norm on (M_1, F_1) .

Theorem 11 (Berwald de Rham decomposition). *Let (M, F) be a simply connected, geodesically complete Berwald space. Then (M, F) is smoothly isometric to a product $(M_0 \times \dots \times M_n, \hat{F})$ given by the de Rham decomposition where $i : M \rightarrow M_0 \times \dots \times M_n$ is the de Rham isometry and $\hat{F} = i_* F$. Furthermore, (M_0, F_0) is a Minkowski space and a curve in (M, F) is a geodesic iff each of its projection is a geodesic.*

If the de Rham decomposition has no factor which is a higher rank symmetric space then the Finsler norm \hat{F} is given by

$$\hat{F}(v_0, \dots, v_n) = G(v_0, F_1(v_1), \dots, F_n(v_n))$$

where $G : \mathbb{R}^{l+n} \rightarrow [0, \infty)$ is a Minkowski norm on $\mathbb{R}^{\dim M_0 + n}$ which is symmetric in the last n -factors.

Remark. This is a more explicit statement of Szabó Berwald classification theorem [Sza06, Theorem 1.3]. Note, however, that the product structure is uniquely determined by a holonomy invariant Minkowski norm on a single tangent space $T_x M = \oplus_i T_{x_i} M_i$.

Proof. Let ∇ denote the induced connection of (M, F) . Then ∇ splits into irreducible components

$$\nabla = \sum_{j=0}^n \nabla_j$$

where ∇_0 is the sum of one dimensional connections and for $j \geq 1$ the connections ∇_j are irreducible. Let (M, g) be the affinely equivalent Riemannian manifold associated to (M, F) . We apply the de Rham Decomposition Theorem to (M, g) , i.e. (M, g) is isometric to a product $(M_0 \times \cdots \times M_n)$ such that M_0 is the maximal Euclidean factor and each (M_i, g_i) is irreducible with connection $(\pi_i)_* \nabla_i$, for simplicity also denoted by ∇_i . Now each factor represents a family of simply connected, totally geodesic submanifolds which are isometric to (M_i, g_i) . As (M, F) and (M, g) are affinely equivalent and (M, g) isometric to $(M_0 \times \cdots \times M_n, \sum g_i)$ we can push forward the Finsler norm to the product. In particular, we can equip each M_i with a Finsler norm such that (M_0, F_0) is a Minkowski space and (M_i, F_i) an irreducible Berwald space whose connections are ∇_i . As auto-parallel curves of (M, F) and (M, g) agree, we see that geodesics are given by geodesic of the factors.

It remains show that the Finsler norm has the given form if none of the factors is a higher rank symmetric space. Note that in this case each (M_i, F_i) is affinely rigid and each F_i is induced by a Riemannian metric.

Now for fixed $v \in V_0$ the functions

$$(w_1, \dots, w_n) \mapsto F(v, w_1, \dots, w_n)$$

are invariant under parallel transport in each coordinate. Now using Theorem 10 we obtain by induction a function $G : \mathbb{R}^{k+n}$ such that

$$F(v, w_1, \dots, w_n) = G(v, F_1(w_1), \dots, F_n(w_n)).$$

Furthermore, we see that F is a Minkowski norm iff G is.

Note that the map $i_F : (M, F) \rightarrow (M_0 \times \cdots \times M_n, F)$ defined via de Rham decomposition and reassigning the Finsler norm is obviously an isometry. In particular, it is a diffeomorphism (use the fact that affine equivalence is a diffeomorphism and then i_F is the de Rham isometry combined with two affine equivalences, another way is to use the results of [DH02]). \square

It is not difficult to see that in case M does not contain a higher rank symmetric factor, the distance on M induced by the Finsler structure that has the following form

$$d_M((x_0, \dots, x_n), (y_0, \dots, y_n)) = G(x_1 - x_0, d_{M_1}(x_1, y_1), \dots, d_{M_n}(x_n, y_n)).$$

In this case we say that the metric space (M, d_M) is a metric product of a Minkowski space and n metric space (M_i, d_{M_i}) .

For Berwald spaces containing a higher rank symmetric factor this is in general not true because there are several functions $v_1 \mapsto G(v_1)$ invariant under the holonomy group which are not given as $\varphi(F_1(v_1))$. In particular, in general

$$F(v_1, v_2) \neq G(F_1(v_1), v_2).$$

Note that if F is C^2 away from the zero section then equality can only hold iff F_1 is C^2 at $v_1 = 0$. But then F_1 is necessarily Riemannian.

4. FLAG AND RICCI CURVATURE

Given an affine connection ∇ one can define the (Riemann) curvature tensor as

$$R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V, W]} Z$$

where V, W, Z are vector fields. If ∇ is the Levi-Civita connection of (M, g) then the sectional curvature spanned by V and W is defined as

$$\mathcal{K}(V, W) = \frac{g(R(V, W)W, V)}{g(V, V)g(W, W) - g(V, W)^2}$$

and the Ricci curvature

$$\text{Ric}(V) = \sum_{i=1}^{n-1} \mathcal{K}(V, e_i)$$

were $\{e_i\}_{i=1}^n$ is an orthonormal basis at $T_x M$ with $e_n = V/F(V)$. Equivalently, one can write the Ricci curvature of a vector in tensor form as follows

$$\text{Ric}(V^i \partial_i) = R_{jik}^i V^k V^k$$

where R_{jik}^i denotes the curvature tensor in local coordinates. Note that this does not involve the metric tensor (g_{ij}) . In particular, it only depends on the connection ∇ .

Instead of now defining the curvatures directly from the Finsler structure we use an interpretation of the flag and Ricci curvature by Shen (see [She01, She97]). Assume the integral curves of the vector field V are non-constant constant speed geodesics. In this case we say V is locally a geodesic field. Then flag curvature $\mathcal{K}^V(V, W)$ spanned by V and W with flag pole V is just the sectional curvature of $g(V)$ spanned by V and W . In a similar way one obtains the Ricci curvature $\text{Ric}(V)$ as the trace of the curvature tensor of $g(V)$. One can actually show that those quantities are defined on the tangent space, i.e. given other geodesic fields V' and W' with $V_x = V'_x$ and $W_x = W'_x$ then $\mathcal{K}^V(V, W)(x) = \mathcal{K}^{V'}(V', W')(x)$ and $\text{Ric}(V)(x) = \text{Ric}(V')(x)$.

A Finsler manifold (M, F) has flag curvature bounded from below by K if

$$\mathcal{K}^V(V, W) \geq K$$

for all unit vector fields V and W . Similarly, the Ricci curvature is bounded from below by K if

$$\text{Ric}(v) \geq K F(v)^2$$

for unit vectors fields V . It is easy to see that a lower bound K on the flag curvature implies the same bound on the Ricci curvature.

A totally geodesic submanifold has the same (possibly better) flag curvature bound as the containing manifolds. For the Ricci curvature this might not be true in general. However, we have the following simplification (see [She01, Prop. 6.6.2]).

Lemma 12. *Let (M, F) be a Berwald space with induced connection ∇ then*

$$\mathcal{K}^V(V, W) = \frac{g_V(R(V, W)W, V)}{g_V(V, V)g(W, W) - g_V(V, W)^2},$$

where R is the curvature tensor of ∇ and $g_V = g(V)$.

Theorem 13 (Ricci affine invariance). *The Ricci curvature of a Berwald space is an affine invariant and only depends on the induced connection, i.e. if (M, F) and (M, F') are affinely equivalent then*

$$\text{Ric}_F(v) = \text{Ric}_{F'}(v).$$

In particular, having non-positive, resp. non-negative Ricci curvature is an affine invariant.

Remark. This was implicitly used in [DKY15, proof of Lemma 4] without explicitly stating the affine invariance.

Proof. Just note that the Ricci curvature is a non-metric contraction of the curvature tensor. In particular, for $v = v^i \partial_i$

$$\text{Ric}_{g_v}(v) = R_{jik}^i(g_v) v^j v^k.$$

However, from the previous lemma,

$$R_{jik}^i(g_v) v^j v^k = R_{jik}^i v^j v^k.$$

□

Remark. Affine invariance of the Ricci curvature holds more general, i.e. if two Finsler structure have the same Chern connection then their Ricci curvatures agree. Indeed, if one denotes by R_{jlk}^i the curvature coefficients of the Chern connection then the arguments above can be used without any change.

With the help of this we can easily show that higher rank symmetric spaces are either compact and have positive Ricci curvature, or they are non-compact and have negative Ricci curvature. Indeed, any symmetric space (M, g) admits a Riemannian metric, namely the Killing form, such that (M, g) is a Einstein space, i.e. $\text{Ric} = \lambda g$. If (M, g) is not flat then $\lambda \neq 0$. In case, $\lambda > 0$ Myers' theorem implies (M, g) is compact. Non-compactness in case $\lambda < 0$ follows by showing that there cannot be closed geodesics.

Lemma 14. *A higher rank symmetric Berwald space of non-negative Ricci curvature has strictly positive Ricci curvature and is compact.*

Finally, some small applications to rigidity of the Berwald condition.

Lemma 15. *Assume (M, F) is a Berwald spaces whose connection splits into two (possibly reducible) factors M_1 and M_2 then M cannot have strictly positive, resp. strictly negative flag curvature at a given flag pole, i.e. $\mathcal{K}^V(V, W) = 0$ for all $V \neq 0$ and some non-zero $W \neq V$.*

Proof. Just note that the curvature tensor R acts trivial on mixed vectors, i.e.

$$R(V_1, V_2) = 0$$

if V_1 is tangent to the M_1 -factor and V_2 tangent to the M_2 -factor. Therefore,

$$\mathcal{K}^{V_1}(V_1, V_2) = \frac{g_{V_1}(R(V_1, V_2)V_2, V_1)}{g_{V_1}(V_1, V_1)g_{V_1}(V_2, V_2) - g_{V_1}(V_1, V_2)^2} = 0.$$

□

Lemma 16. *If (M, F) is an irreducible higher rank symmetric Berwald space then it cannot have strictly its flag curvature strictly positive, resp. strictly negative flag curvature at a given flag pole.*

Proof. Through every point there is an at least two dimensional totally geodesic flat manifolds. The flag curvature to this submanifold is zero, hence prohibiting any strict bounds. \square

Combing these facts we see that any strictly curved space must be irreducible and cannot be a higher rank symmetric space. By Theorem 10 it must be Riemannian.

Corollary 17. *Let (M, F) be a Berwald space. If its flag curvature (see below) is strictly positive, resp. negative then it is irreducible and Riemannian.*

This can now be used to show that there are no strictly curved symmetric Finsler spaces. This simple argument was already used by Matveev [Mat15].

Corollary 18. *Let (M, F) be an irreducible (locally) symmetric Finsler space with strictly positive (resp. strictly negative) flag curvature. Then (M, F) is Riemannian.*

Proof. Just note that every locally symmetric Finsler space is locally Berwald (see [MT12, Theorem 9.2]). Combining with the above we obtain that the space is Riemannian. \square

5. SPLITTING THEOREMS

The splitting theorem is now an easy consequence of the Berwald de Rham decomposition (Theorem 11) and affine invariance of the Ricci curvature (see Theorem 13). First recall the Cheeger-Gromoll splitting theorem.

Lemma 19 (Cheeger-Gromoll Splitting Theorem). *Let (M, g) be a geodesically complete Riemannian manifold of non-negative Ricci curvature containing an isometrically embedded line $\eta : \mathbb{R} \rightarrow M$. Then (M, g) is isometric to a Cartesian product $(M' \times \mathbb{R}, g' + (\cdot, \cdot))$ where g' induces a Riemannian metric on M' whose Ricci curvature is non-negative. In particular, the Levi-Civita connection of M contains a flat one dimensional factor acting on the \mathbb{R} -factor.*

Theorem 20. *Let (M, g) be a simply connected geodesically complete Riemannian with non-negative Ricci curvature. Then M is isometric to a product $(M_0 \times \cdots \times M_n, \sum g_i)$ where M_0 is a flat Euclidean space and each (M_i, g_i) is a simply connected irreducible Riemannian manifold of non-negative. Furthermore, each line contained in M is tangent to the M_0 -factor and has constant M_i -coordinate.*

Proof. This follows by combining the de Rham decomposition and the Cheeger-Gromoll splitting Theorem above. Indeed, each irreducible factor needs to have non-negative Ricci curvature. If there is a line not constant on a factor M_i then there is actually a line only tangent to the M_i -factor. But then the connection on M_i is reducible which is impossible. \square

Note that if M is not simply connected and the fundamental group is torsion-free then each element in the fundamental group is represented by a closed geodesic. But this geodesic lifts to a line in the universal cover which has non-negative Ricci curvature as well.

Corollary 21. *A geodesically complete Riemannian manifold (M, F) with torsion-free fundamental group and non-negative Ricci curvature splits into a product as above where M_0 can be of the form $\mathbb{R}^k \times \mathbb{T}^l$. Furthermore, every non-flat factor is simply connected.*

Now the Berwald splitting follows immediately.

Theorem 22. *Let (M, F) be a simply connected geodesically complete Berwald space of non-negative Ricci curvature. Then M is a product $M = M_0 \times \cdots \times M_n$ where $M_0 = \mathbb{R}^k$ is flat and each (M_i, F_i) is simply connected and either an irreducible Riemannian manifold of non-negative Ricci curvature or a higher rank symmetric Berwald space of compact type.*

Again, if none of the factors is a higher rank symmetric space then one obtains a more explicit form of the Finsler metric. Furthermore, in the torsion-free case one can actually split off all non-flat factors.

Since every symmetric space is Einstein one might ask whether a similar statement holds for higher rank symmetric Berwald spaces. More general, what is the general structure of Berwald-Einstein spaces.

Definition 23 (Berwald-Einstein). A Berwald space (M, F) is called Berwald-Einstein space if for every $v_x \in TM$

$$\text{Ric}(v_x) = \lambda(x)F(v_x)^2,$$

where λ is a (fixed) function on M .

We say that (M, F) is Ricci-flat, if $\text{Ric} = 0$. Note that any Ricci-flat Berwald space is Berwald-Einstein. Furthermore, it is known that if (M, F) is Riemannian then λ needs to be constant.

Lemma 24 (Berwald-Einstein Rigidity [DKY15]). *A connected Berwald-Einstein space is either Ricci-flat or a Riemannian manifold. In each case λ is constant.*

Proof. Let $U = \{x \in M \mid \lambda(x) \neq 0\}$. Note that Ric is a quadratic form. Suppose $U \neq \emptyset$ then $F|_U$ is a quadratic form as well. In particular, it is induced by an inner product. But as F is preserved by parallel transport also $F|_{M \setminus U}$ must be induced by an inner product. In particular, (M, F) is Riemannian. \square

Theorem 25 (Berwald-Einstein Structure Theorem). *A geodesically complete Berwald-Einstein space (M, F) is either an Einstein (Riemannian) manifold or it is Ricci-flat and each non-flat factor M_i is an irreducible Ricci-flat Riemannian manifold and F is given as in Theorem 11.*

Proof. If (M, F) is not Ricci-flat then it is Riemannian. In the Ricci-flat case, note that a higher rank symmetric Berwald space has non-zero Ricci curvature as there is an affinely equivalent non-flat Einstein metric on such a space. Therefore, the full structure theorem applies. \square

Because non-negative flag curvature implies non-negative Ricci curvature we can apply the soul theorem to each of the factors and obtain a weaker version of this theorem for Berwald spaces.

Proposition 26 (Berwald soul theorem). *Assume (M, F) is a geodesically complete Berwald space of non-negative flag curvature. Then there is a compact totally convex, totally geodesic submanifold S , called soul of M , such that M is diffeomorphic to the normal bundle of S .*

Remark. We only show the existence of the soul. It is unclear whether the Sharafutdinov map [Sha78] is also distance non-increasing in this setting. The contractive behavior of that map is proved via a gradient flows of convex functions. In the

Finsler setting, even on flat Minkowski spaces, the gradient flow of convex function is not necessarily contractive (see [OS12, Theorem 3.2]).

Proof. The proof follows by combining the splitting theorem above and the soul theorem applied to each irreducible Riemannian factor of the splitting.

Namely, non-negative flag curvature implies that M has non-negative Ricci curvature. By the previous theorem $M = M_0 \times \cdots \times M_n$ where (M_i, F_i) are irreducible Riemannian or higher rank symmetric spaces. Because (M, F) has non-negative flag curvature, so does every Riemannian factor $(M_i, F_i) = (M_i, g_i)$. For every higher rank symmetric factor (M_i, F_i) , note that every affinely equivalent Einstein metric g_i on M_i has non-negative sectional curvature.

Take the Cartesian product $(M, \sum g_i)$ of the Riemannian metrics (with g_0 being any inner product on M_0). Then there is a compact totally convex, totally geodesic Riemannian submanifold S of $(M, \sum g_i)$. Because (M, F) and $(M, \sum g_i)$ are affinely equivalent, S is also a compact totally convex, totally geodesic submanifold of (M, F) . Furthermore, the soul theorem shows that M is diffeomorphic to the normal bundle of S . If one uses the forward exponential map on (M, F) instead of the exponential map of (M, g) then this diffeomorphism is on C^2 and C^∞ iff (M, F) is actually a Riemannian manifold and F induced by g . \square

As an application we can show a conjecture by Lakzian [Lak14] on non-negatively curved reversible Berwald spaces.

Corollary 27. *A geodesically complete Berwald space (M, F) of non-negative flag curvature with large volume growth is diffeomorphic to \mathbb{R}^n .*

Proof. Using the previous preposition we see that (M, F) is diffeomorphic to the normal bundle of a compact totally convex, totally geodesics submanifold S . If S is not a point then it contains a closed geodesic, and then also (M, F) . Indeed, (S, F_S) is affinely equivalent to a geodesically complete, compact Riemannian manifold (S, g_S) . Because (S, g_S) is Riemannian it contains a closed geodesics (see [Kli95]).

However, Lakzian showed [Lak14] that a non-negatively curved Berwald space with large volume growth cannot contain any closed geodesics. Therefore, S is a point and M diffeomorphic to \mathbb{R}^n .

Note that the assumption on reversibility is not needed. Indeed, if (M, F, \mathbf{m}) has large volume growth (see [Lak14, 3.2]) then so has (M, g, \mathbf{m}) where (M, g) is affinely equivalent to (M, F) and has non-negative sectional curvature. \square

Remark. Lakzian actually showed the non-existence of closed geodesics for Berwald spaces of non-negative radial flag curvature with large volume growth. If non-negative radial flag curvature of (M, F) implies non-negative sectional curvature of an affinely equivalent Riemannian manifold (M, g) then the main result of [Lak14] can be reduced to the Riemannian setting as well. It is likely that this follows if one shows that every higher rank symmetric Berwald space has the same curvature bound as its Riemannian equivalent.

6. WEIGHTED RICCI CURVATURE

Similar to Riemannian manifolds there is a Finsler version of *weighted Ricci curvature*, see [Oht09]. However, instead of the general version we will use a Berwald version resembling the weighted Ricci curvature for Riemannian manifolds. With

this we can apply again the structure theorem for Riemannian manifold with non-negative weighted Ricci curvature.

Let us first define the weighted Ricci curvature: Denote by vol_F the *Busemann-Hausdorff measure*. One can show that this measure is a multiple of the volume form vol_g of an affinely equivalent Riemannian structure g on M (follows from the proof of [She01, Prop. 7.3.1]).

Now we can equip any Berwald space with a smooth measure \mathbf{m} which is absolutely continuous w.r.t. vol_F . Then there is a function $\Psi : M \rightarrow \mathbb{R}$ such that

$$\mathbf{m} = e^{-\Psi} \text{vol}_F.$$

Definition 28 (Weighted Ricci curvature). Denote by n the dimension of M and let $N' \in (n, \infty)$ and $v = \dot{\eta}(0)$ be a tangent vector. We define the following quantities:

- $\text{Ric}_n(v) = \begin{cases} \text{Ric}(v) + (\Psi \circ \eta)''(0) & \text{if } (\Psi \circ \eta)'(0)=0 \\ -\infty & \text{if } (\Psi \circ \eta)'(0) \neq 0 \end{cases}$
- $\text{Ric}_{N'}(v) = \text{Ric}(v) + (\Psi \circ \eta)''(0) + \frac{((\Psi \circ \eta)'(0))^2}{N'-n}$
- $\text{Ric}_\infty(v) = \text{Ric}(v) + (\Psi \circ \eta)''(0).$

Note that we have $\text{Ric}_N(cv) = c^2 \text{Ric}_{N'}$ for $N \in [n, \infty]$ and $c > 0$.

Now we say that (M, F, \mathbf{m}) the N -dimensional Ricci curvature bounded from below if

$$\text{Ric}_N(v) \geq K F(v)^2.$$

Remark. If F is induced by a Riemannian metric then it is exactly the definition of weighted (Riemannian) Ricci curvature. Furthermore, one can show that it is equivalent to Ohta's original Finsler definition, see [Oht09, Oht13]. Indeed, first note that there is a function $\Phi : M \rightarrow \mathbb{R}$ such that

$$\text{vol} = e^{-\Phi} \text{vol}_{\dot{\eta}}$$

where $\text{vol}_{\dot{\eta}}$ denote the Riemannian volume measure w.r.t. $g(\dot{\eta})$. As vol_F is the Busemann-Hausdorff measure $(\Phi \circ \eta)'(0) \equiv 0$ [She01, Prop. 7.3.1]. Now if

$$\mathbf{m} = e^{-\tilde{\Psi}(\eta)} \text{vol}_{\dot{\eta}}$$

for some function $\tilde{\Psi} : M \rightarrow \mathbb{R}$ then $\Psi = \tilde{\Psi} - \Phi$. But then $(\tilde{\Psi} \circ \eta)'(0) = (\Psi \circ \eta)'(0)$ and $(\tilde{\Psi} \circ \eta)''(0) = (\Psi \circ \eta)''(0)$. In particular, the definitions of weighted Ricci curvatures agree.

Proposition 29. *Weighted Ricci curvature is an affine invariant.*

Proof. Let (M, F) and (M, F') be affinely equivalent. Equip M with a measure \mathbf{m} . Then we have the decomposition

$$\mathbf{m} = e^{-\Phi_F} \text{vol}_F = e^{-\Phi_{F'}} \text{vol}_{F'}.$$

As the Busemann-Hausdorff measures are multiples of each other we must have $\Phi_F = \Phi_{F'} + c$ for some constant $c \in \mathbb{R}$. But geodesics as curves agree so that we must have $\text{Ric}_N^F = \text{Ric}_N^{F'}$. \square

Similar to the discussion above we get the following the structure theorem for non-negative weighted Ricci curvature: By an isometrically embedded line $\eta : \mathbb{R} \rightarrow M$ we implicitly equip \mathbb{R} either with a symmetric metric or an asymmetric metric to allow for lines inside the Minkowski factor.

Lemma 30 (Weighted Berwald Splitting Theorem). *Let (M, F, \mathbf{m}) be a geodesically complete Berwald space with non-negative N -Ricci curvature. If M contains an isometrically embedded line then there is a measure preserving isometry onto $(\mathbb{R} \times M', F', \lambda \times \mathbf{m}')$.*

Theorem 31. *Let (M, F, \mathbf{m}) be a geodesically complete Berwald space with non-negative N -Ricci curvature. Then there is a measure-preserving isometry onto $(M_0 \times M', \hat{F}, \lambda^k \mathbf{m}')$ where $M_0 = \mathbb{R}^k$ is flat, λ^k is the Lebesgue measure on \mathbb{R}^k and $(M', \hat{F}_{M'}, \mathbf{m}')$ has non-negative $(N - k)$ -dimensional Ricci curvature. In particular, all lines are entirely tangent to the flat factor. Furthermore, none of the factors of M' is a higher rank symmetric space of non-compact type.*

Remark. In contrast to the general structure theorem it is not possible to split of all flat factors as it is not clear that all but the flat factors of M' are simply connected.

Similar to the structure theorem the Finsler norm can be given more explicitly if none of the factors of M' is a higher rank symmetric Berwald space of compact-type.

7. CONCLUSION AND QUESTIONS

We have shown that excluding symmetric factors, the topology and many geometric properties reduce to the study of metric products of irreducible Riemannian manifolds. Furthermore, (weighted) Ricci curvature is an affine invariant in the category of Berwald spaces. In particular, non-negative/non-positive Ricci curvature is an affine invariant. Since non-negative N -Ricci curvature is equivalent to Lott-Sturm-Villani's curvature dimension condition $CD(0, N)$ (see [Oht09] for definition and proof for Finsler manifolds), one might ask whether this holds more generally.

Question 32. *Assume (X_i, d_i, \mathbf{m}_i) are metric spaces satisfying the $CD(0, N_i)$ -condition. Is it true that the metric product $(X_1 \times \dots \times X_n, d, \times_i \mathbf{m}_i)$ with*

$$d(\mathbf{x}, \mathbf{y}) = F(d(x_1, y_1), \dots, d(x_n, y_n))$$

satisfies the $CD(0, \sum N_i)$ -condition?

Easier to answer might be the question whether this holds for the metric products of RCD -spaces. More general one could ask whether $CD(K_i, N_i)$ implies $CD(K, N)$ for some K .

The concept of affine equivalence can be defined also for geodesic metric spaces. Namely, two metrics on X are affinely equivalent if every geodesic of (X, d_1) is a geodesic of (X, d_2) .

Question 33. *Assume (X, d) is an $RCD(0, N)$ -space. Does every affinely equivalent metric space (X, d') satisfy $CD(0, N)$?*

It is not difficult to see that this is true in the compact setting if the definition of $CD(0, N)$ is relaxed to allow convexity of the Renyi entropy to hold along “some” curve in the Wasserstein space $\mathcal{P}_2(X, d)$, compare [AGS08, 9.2].

REFERENCES

- [AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, second ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [CG71] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, Journal of Differential Geometry **6** (1971), no. 1, 119–128.

- [DH02] S Deng and Z Hou, *The group of isometries of a Finsler space*, Pacific Journal of Mathematics **207** (2002), no. 1, 149–155.
- [DKY15] S. Deng, D. C. Kertesz, and Z. Yan, *There are no proper Berwald–Einstein manifolds*, Publicationes Mathematicae Debrecen **86** (2015), 245–249.
- [dR52] G. de Rham, *Sur la réductibilité d'un espace de Riemann*, Commentarii Mathematici Helvetici **26** (1952), no. 1, 328–344.
- [KK06] A. Kristály and L. Kozma, *Metric characterization of Berwald spaces of non-positive flag curvature*, Journal of Geometry and Physics **56** (2006), no. 8, 1257–1270.
- [Kli95] W. Klingenberg, *Riemannian Geometry*, Walter de Gruyter, 1995.
- [Lak14] S. Lakzian, *On Closed Geodesics in Non-negatively Curved Finsler Structures with Large Volume Growth*, arxiv:1408.0214 (2014), 14.
- [Mat15] V. S. Matveev, *There exist no locally symmetric Finsler spaces of positive or negative flag curvature*, Comptes Rendus Mathématique **353** (2015), no. 1, 81–83.
- [MT12] V. S. Matveev and M. Troyanov, *The Binet-Legendre Metric in Finsler Geometry*, Geometry & Topology **16** (2012), no. 4, 2135–2170.
- [Oht09] S. Ohta, *Finsler interpolation inequalities*, Calculus of Variations and Partial Differential Equations **36** (2009), no. 2, 211–249.
- [Oht13] ———, *Splitting theorems for Finsler manifolds of nonnegative Ricci curvature*, Journal für die reine und angewandte Mathematik (2013).
- [Olm05] C. Olmos, *A geometric proof of the Berger Holonomy Theorem*, Annals of Mathematics **161** (2005), no. 1, 579–588.
- [OS12] S. Ohta and K.-Th. Sturm, *Non-Contraction of Heat Flow on Minkowski Spaces*, Archive for Rational Mechanics and Analysis **204** (2012), no. 3, 917–944.
- [Per94] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, Journal of Differential Geometry **40** (1994), no. 1, 209–212.
- [Sha78] V. A. Sharafutdinov, *The Pogorelov-Klingenberg theorem for manifolds homeomorphic to R^n* , Siberian Mathematical Journal **18** (1978), no. 4, 649–657.
- [She96] Z. Shen, *Finsler manifolds of constant positive curvature*, Contemporary Mathematics **196** (1996), 83–96.
- [She97] ———, *Curvature, Distance and Volume in Finsler Geometry*, preprint (1997).
- [She01] ———, *Lectures on Finsler Geometry*, World Scientific, 2001.
- [She02] ———, *Two-dimensional Finsler metrics with constant flag curvature*, manuscripta mathematica **109** (2002), no. 3, 349–366.
- [SLK11] J. Szilasi, R.L. Lovas, and D.Cs. Kertész, *Several ways to a Berwald manifold - and some steps beyond*, Extracta Mathematicae **26** (2011), no. 1.
- [Sza81] Z. I. Szabó, *Positive definite Berwald spaces*, Tensor N. S. **35** (1981), 25–39.
- [Sza06] ———, *Berwald metrics constructed by Chevalley's polynomials*, arxiv:0601522 (2006), 28.

E-mail address: `mkell@ihes.fr`

INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, F-91440 BURES-SUR-YVETTE, FRANCE